The van Hemmen's Ashkin-Teller spin glass

1992 J. Phys. A: Math. Gen. 25 L739

(http://iopscience.iop.org/0305-4470/25/12/006)

J. Phys. A: Math. Gen. 25 (1992) L739-L744. Printed in the UK

LETTER TO THE EDITOR

The van Hemmen's Ashkin–Teller spin glass

J V Moreira† and P L Christiano‡

† Departamento de Matemática, Universidade Federal da Paraíba, 58000 João Pessoa PB, Brazil

‡ Departamento de Física, Universidade Federal da Paraíba, 58000 João Pessoa PB, Brazil

Received 9 December 1991

Abstract. We study a van Hemmen's version of the infinite-ranged Ashkin-Teller spin glass with particular emphasis on the determination of its phase diagram. Comparisons with the Sherrington-Kirkpatrick version of the same model and with the random bond Ashkin-Teller model on the Bethe lattice are also made.

In a recent paper (Moreira and Christiano 1991) we studied the infinite-ranged Ashkin-Teller spin glass with Gaussian distributed exchange interactions as proposed by Sherrington and Kirkpatrick (1975) and Kirkpatrick and Sherrington (1978) for the Ising model. In this letter we adopt a complementary approach and study the Ashkin-Teller spin glass following the model introduced by van Hemmen (1982) and van Hemmen *et al* (1983) (hereafter referred to as VH) several years ago. This model, besides allowing a simple mean field description of the spin glasses, offers a quite direct application to neural networks such as the Hopfield model (Hopfield 1982). In that sense, the extension of this study to the Ashkin-Teller model (ATM) (Ashkin and Teller 1943) enables the introduction of neural networks with two levels of memory which seems to us very interesting and promising.

As is well known (Fan 1972), the ATM Hamiltonian can be written using Ising spin variables by putting two of these variables in each site and introducing a four-body interaction, so that the infinite-ranged Ashkin-Teller spin-glass Hamiltonian should be written as

$$H = -\tfrac{1}{2}\sum_{\langle ij\rangle}\left[J_{ij}^{(1)}\sigma_i\sigma_j + J_{ij}^{(2)}S_iS_j + J_{ij}^{(3)}\sigma_iS_i\sigma_jS_j\right] \tag{1}$$

where the σ's and the S's are Ising spin variables assuming only the values ± 1, the J's are random interactions and the sum runs over every pair of spins. Following Provost and Vallee (1983) we write the van Hemmen version of the ATM spin-glass Hamiltonian in the more general form:

$$H = -\frac{1}{N}\sum_{\langle ij\rangle}\left[J_1^0\sigma_i\sigma_j + J_2^0S_iS_j + J_3^0\sigma_iS_i\sigma_jS_j\right]$$
$$-\frac{1}{N}\sum_{\langle ij\rangle}\left[\sum_{\mu\nu}J_1^{\mu\nu}\xi_{i\mu}\xi_{j\nu}\sigma_i\sigma_j + \sum_{\alpha\beta}J_2^{\alpha\beta}\gamma_{i\alpha}\gamma_{j\beta}S_iS_j + \sum_{\theta\epsilon}J_3^{\theta\epsilon}\psi_{i\theta}\psi_{j\epsilon}\sigma_iS_i\sigma_jS_j\right] \tag{2}$$

where J_1^0, J_2^0 and J_3^0 are ferromagnetic couplings and the ξ's, γ's and ψ's are random variables distributed according to the law:

$$P(x) = \tfrac{1}{2}\delta(x-1) + \tfrac{1}{2}\delta(x+1). \tag{3}$$

0305-4470/92/120739+06$04.50

A direct adaptation of the original VH proposition is obtained when the sums over μ, γ, α, β, θ and ε run from 1 to 2 and the matrices $\mathbf{J}_i$ $(i=1,2,3)$ are defined as:

$$\mathbf{J}_i=\begin{bmatrix}0 & J_i\\ J_i & 0\end{bmatrix} \qquad i=1,2,3. \tag{4}$$

It is worth mentioning that this is only one of the possible generalizations of the VH spin glass for the ATM. Another quite different possibility, for instance, is obtained by assuming that the ψ's are not independent variables. In this case, they can be constructed as functions of the ξ's and γ's.

Introducing the vectorial notation $\boldsymbol{\xi}_i=(\xi_{i1},\xi_{i2},\dots)$, $\boldsymbol{\gamma}_i=(\gamma_{i1},\gamma_{i2},\dots)$ and $\boldsymbol{\psi}_i=(\psi_{i1},\psi_{i2},\dots)$ and defining:

$$m_1=\frac{1}{N}\sum_i \sigma_i \qquad \boldsymbol{q}_1=\frac{1}{N}\sum_i \boldsymbol{\xi}_i\sigma_i \tag{5a}$$

$$m_2=\frac{1}{N}\sum_i S_i \qquad \boldsymbol{q}_2=\frac{1}{N}\sum_i \boldsymbol{\gamma}_i S_i \tag{5b}$$

$$m_3=\frac{1}{N}\sum_i \sigma_i S_i \qquad \boldsymbol{q}_3=\frac{1}{N}\sum_i \boldsymbol{\psi}_i\sigma_i S_i \tag{5c}$$

the Hamiltonian (2) can be written as

$$H=-\frac{N}{2}[J_1^0m_1^2+J_2^0m_2^2+J_3^0m_3^2+\boldsymbol{q}_1\mathbf{J}_1\boldsymbol{q}_1+\boldsymbol{q}_2\mathbf{J}_2\boldsymbol{q}_2+\boldsymbol{q}_3\mathbf{J}_3\boldsymbol{q}_3]. \tag{6}$$

From this Hamiltonian the procedure introduced by Provost and Vallee can be straightforwardly generalized. Then, using the Gaussian identity

$$\exp[\lambda a^2]=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}x\,\exp[-(x^2/2)+(2\lambda)^{1/2}ax] \tag{7}$$

performing the trace and using steepest descent integrations one obtains the free energy per spin:

$$f=\tfrac{1}{2}[J_1^0m_1^0+J_2^0m_2^2+J_3^0m_3^2+\boldsymbol{q}_1\mathbf{J}_1\boldsymbol{q}_1+\boldsymbol{q}_2\mathbf{J}_2\boldsymbol{q}_2+\boldsymbol{q}_3\mathbf{J}_3\boldsymbol{q}_3]$$
$$-\frac{1}{\beta}\langle\ln\{4\cosh L_1\cosh L_2\cosh L_3(1+\tanh L_1\tanh L_2\tanh L_3)\}\rangle \tag{8}$$

where

$$L_1=\beta J_1^0m_1+\beta\boldsymbol{\xi}\mathbf{J}_1\boldsymbol{q}_1 \tag{9a}$$

$$L_2=\beta J_2^0m_2+\beta\boldsymbol{\gamma}\mathbf{J}_2\boldsymbol{q}_2 \tag{9b}$$

$$L_3=\beta J_3^0m_3+\beta\boldsymbol{\psi}\mathbf{J}_3\boldsymbol{q}_3. \tag{9c}$$

The order parameters m_α, q_α $(\alpha=1,2,3)$ are obtained from this free energy imposing the extremal conditions

$$(\partial f/\partial m_\alpha)=0 \qquad (\partial f/\partial q_\alpha)=0 \qquad (\alpha=1,2,3) \tag{10}$$

as

$$m_\alpha=\langle(\tanh L_\alpha+\tanh L_\beta\tanh L_\gamma)/(1+\tanh L_\alpha\tanh L_\beta\tanh L_\gamma)\rangle \tag{11a}$$

$$q_\alpha^{(\mu)}=\langle\theta_\mu(\tanh L_\alpha+\tanh L_\beta\tanh L_\gamma)/(1+\tanh L_\alpha\tanh L_\beta\tanh L_\gamma)\rangle \tag{11b}$$

where α, β and γ are taken to be different and θ_μ is the corresponding random variable.

Through the introduction of auxiliary external magnetic fields h_1, h_2 and h_3 acting on the σ's, S's and σS's, respectively, it is easy to show that the m's are the usual ferromagnetic order parameters for the ATM

$$m_1 = \langle\langle \sigma \rangle\rangle \qquad m_2 = \langle\langle S \rangle\rangle \qquad m_3 = \langle\langle \sigma S \rangle\rangle \tag{12a}$$

where the internal brackets mean thermal average and the external ones the average over the random variables. Concerning the other order parameters one obtains

$$q_1^\mu = \langle \xi_\mu \langle \sigma \rangle\rangle \qquad q_2^\mu = \langle \gamma_\mu \langle S \rangle\rangle \qquad q_3^\mu = \langle \psi_\mu \langle \sigma S \rangle\rangle. \tag{12b}$$

They can be interpreted as the spin-glass order parameters observing that a non-zero q means that there is a correlation between the average value of the spin variable and a random variable or, in other terms, that this spin is frozen at random.

Specializing for the particular case in which the random variables ξ, γ and ψ, and then also the order parameters q_1^μ, q_2^μ and q_3^μ, have only two components, as originally proposed by van Hemmen, the free energy (8) is reduced to

$$f = \tfrac{1}{2}[J_1^0 m_1^2 + J_2^0 m_2^2 + J_3^0 m_3^2] + J_1 q_1^{(1)} q_1^{(2)} + J_2 q_2^{(1)} q_2^{(2)} + J_3 q_3^{(1)} q_3^{(2)}$$
$$- \frac{1}{\beta} \langle \ln(4 \cosh L_1 \cosh L_2 \cosh L_3)(1 + \tanh L_1 \tanh L_2 \tanh L_3) \rangle. \tag{13}$$

Following VH, we now assume that the extremum of the free energy is obtained when the two components of the order parameters q_α ($\alpha = 1, 2, 3$) are equal so that it can be written as

$$f = \tfrac{1}{2}[J_1^0 m_1^2 + J_2^0 m_2^2 + J_3^0 m_3^2] + J_1 q_1^2 + J_2 q_2^2 + J_3 q_3^2$$
$$- \frac{1}{\beta} \langle \ln\{4 \cosh L_1 \cosh L_2 \cosh L_3 (1 + \tanh L_1 \tanh L_2 \tanh L_3)\} \rangle \tag{14}$$

and the order parameters as

$$m_\alpha = \langle (\tanh L_\alpha + \tanh L_\beta \tanh L_\gamma)/(1 + \tanh L_\alpha \tanh L_\beta \tanh L_\gamma) \rangle \tag{15a}$$

$$q_\alpha = \langle \tfrac{1}{2}(\theta_1 + \theta_2)(\tanh L_\alpha + \tanh L_\beta \tanh L_\gamma)/(1 + \tanh L_\alpha \tanh L_\beta \tanh L_\gamma) \tag{15b}$$

where, now

$$L_\alpha = \beta J_\alpha^0 m_\alpha + \beta J_\alpha q_\alpha (\theta_1 + \theta_2) \tag{16}$$

where we are using the notation introduced in (11). As those equations involve an excessive number of parameters, in the following we will restrict ourselves to the study of some few particular cases in which there are only two parameters. These particular cases are obtained either by the vanishing of some of the parameters or by imposing some constraints between them.

Proceeding in this way we observe that the VH Ising spin glass in the variable σS is recovered when $J_1^0 = J_2^0 = 0$ and $J_1 = J_2 = 0$ and the regular ATM (Christiano and Goulart Rosa 1985) when $J_1 = J_2 = J_3 = 0$. In the other particular cases studied we will always impose that $J_1^0 = J_2^0 = J_0$ and $J_1 = J_2 = J$. As can be easily seen from equations (15) under these constraints, the variables σ and S will display the same type to ordering. In the phase diagrams (figures 1) this ordering is indicated by the first letter while the second one describes the ordering of σS. Also, in these phase diagrams, the dashed lines indicate the regions in which there is more than one solution. For sufficiently high temperatures all phase diagrams present a paramagnetic phase (P-P) in which there is no ordering at all and then all the order parameters vanish. Let us discuss and describe the others phases shown in the phase diagrams.

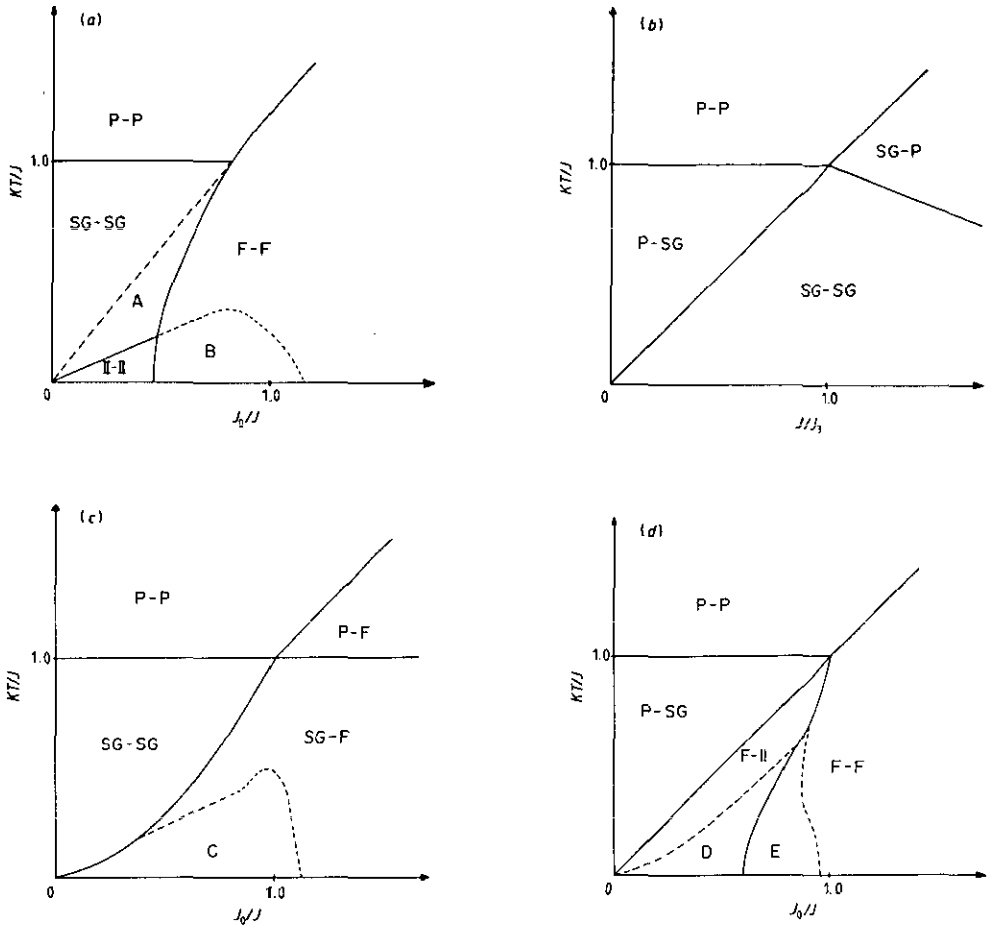

Figure 1. Phase diagrams for the particular cases in which $J_1^0 = J_2^0 = J_0$ and $J_1 = J_2 = J$ and (*a*) $J_3^0 = J_0$, $J_3 = J$; (*b*) $J_3^0 = J_0 = 0$; (*c*) $J_0 = J_3 = 0$; (*d*) $J_3^0 = J = 0$. The phases are indicated by two letters: the first corresponds to the ordering of the individual spins σ and S and the second to the ordering of the pair σS. Besides the usual paramagnetic (P), ferromagnetic (F) and spin glass (SG) orderings a mixed (II) ordering in which all the order parameters are non-zero is also shown. The letters A, B, C, D and E indicate regions of coexistence of solutions.

Case A. $J_1^0 = J_2^0 = J_3^0 = J_0$ and $J_1 = J_2 = J_3 = J$ (figure 1(*a*))

In this case, as can be seen from equations (15), $m_1 = m_2 = m_3 = m$ and $q_1 = q_2 = q_3 = q$. So, the variable σS presents the same type of ordering of σ and S and the complete set of order parameters is reduced to only two—m and q. This observation is confirmed by the numerical analysis and it is also possible to show analytically that Ising-like solutions in which only one pair of order parameters (e.g. m_1 and q_1) is different to zero are unstable everywhere. These solutions should correspond to the Mattis state found in the Hopfield neural network model (Amit *et al* 1985). Besides the paramagnetic phase, there is a spin glass phase (SG-SG) in which $m = 0$ and $q \neq 0$, a ferromagnetic phase (F-F) in which $m \neq 0$ and $q = 0$ and a mixed phase that, following VH, is indicated as (II-II). In this phase $q \neq 0$ indicating that the spins are frozen and random but also $m \neq 0$ indicating that there is a ferromagnetic component, so that one should say that

in this phase the spins are randomly frozen around a preferred direction. In region A the ferromagnetic and spin glass solutions are simultaneously present but the ferromagnetic solution represents only a local minimum of the free energy. In region B and in the region occupied by the mixed phase the coexistence happens to be between this solution and the ferromagnetic one that corresponds to a global minimum only in region *B*. The appearance of these regions in which two solutions coexist is an indicator of first-order phase transitions in the sense that there is a gap in the order parameters.

Case B. $J_3^0 = J_0 = 0$ (figure 1(*b*))

It is easy to see that under these conditions $m_1 = m_2 = m_3 = 0$ and the system of equations (15) is reduced to:

$$q_1 = q_2 = \langle \tfrac{1}{2}(\xi_1 + \xi_2)(\tanh L_1 + \tanh L_2 \tanh L_3)/(1 + \tanh L_1 \tanh L_2 \tanh L_3)\rangle \qquad (17a)$$

$$q_3 = \langle \tfrac{1}{2}(\psi_1 + \psi_2)(\tanh L_1 + \tanh L_2 \tanh L_3)/(1 + \tanh L_1 \tanh L_2 \tanh L_3)\rangle. \qquad (17b)$$

In this case, the phase diagram is quite different from the previous one. First of all it should be noted that there are two phases in which the variable σS does not display the same type of ordering of σ (and S). For $J_3 > KT > J$ the individual spins σ and S do not order at all, behaving as in a paramagnetic phase, while the pair σS assumes a spin-glass ordering. This phase is indicated by P–SG in the phase diagram. The other phase in which the spins do not show the same type or ordering is the phase SG–P in which the individual spins $\sigma(S)$ freeze in a spin-glass phase and the pair σS remains disordered or in a paramagnetic phase. It should be noted that this phase is not restricted to the region $J > KT > J_3$, as expected, but it also penetrates in a region in the phase diagram where $J_3 > KT$. This is a clear indication that the spin-glass ordering of the individual spins $\sigma(S)$ make difficult the spin-glass ordering of the pair σS when $J > J_3$. Another important difference between these phase diagrams is the disappearance of the regions where two or more solutions coexist. Considering only the behaviour of the order parameters, all transitions in this case should be second order.

This phase diagram can be compared with the equivalent ones obtained in the Sherrington–Kirkpatrick version of the Ashkin–Teller spin glass (figure 2(*a*) in Moreira and Christiano (1991)) and with that obtained for the random-bond isotropic ATM in the Bethe lattice (figure 2(*a*) in Christiano and Goulart Rosa (1986)). There are important differences. Concerning the former we observe that the main differences refer to the appearance of the phase SG–P and a dislocation in the transition line between the phase P–SG and SG–SG. It is important to mention that the phase diagram in this case was constructed considering the replica symmetric approach and that all these differences happens to be in the region in which this solution is unstable. Besides this, the spin-glass order parameter in the replica symmetric approach measures the average over the disorder of the spin variables squared thermal averages and then the ordering of the individual spins σ and S, even in a spin-glass phase, drives the ordering of the pair σS, preventing the appearance of a paramagnetic phase concerning this variable. On the other hand, the spin-glass ordering of the pair σS also drives the ordering of the individual spins σ and S in some sense, facilitating the spin-glass ordering of these variables, which accounts for the dislocation of the transition line referred above. One can say that the Ashkin–Teller spin-glass order parameters in the Sherrington–Kirkpatrick replica symmetric approach are more correlated than they are in the VH version in which the replica approach was not used. It is worth mentioning that this absence of correlations between the order parameters in this VH version is an

artifact of its construction considering the random variables associated with the pair σS independent of the ones associated with the individual spins. Of course, when one assumes that these variables are correlated, quite different results are expected. A simple version of this model introducing correlations between these variables will be published elsewhere.

Concerning the random bond isotropic ATM in the Bethe lattice, as observed by Christiano and Goulart Rosa (1986) the main differences between the phase diagrams can be attributed to the fact that when $J_{ij}^{(1)}$ is made equal to $J_{ij}^{(2)}$ the correlations between the order parameters are even greater and the ordering of the individual spins, even in a spin-glass phase, drives the ferromagnetic ordering of the pair σS. In this case, the phases SG–SG and P–SG are substituted by a spin-glass–ferromagnetic phase.

Case C. $J_0 = J_3 = 0$ (figure 1(*c*))

This phase diagram is very similar to that obtained for the Sherrington–Kirkpatrick version of the Ashkin–Teller spin glass, differing only in the localization of the transition line between the SG–SG phase (in which both the individual spins and the pair σS assume a spin-glass ordering) and the SG–F phase (in which the individual spins assume a spin-glass ordering and the pair σS a ferromagnetic ordering). In region C the SG–F solution coexists with a SG–II solution (in which the individual spins present a spin-glass ordering and the pair σS 'orders' in a mixed phase as discussed in case A) but the latter represents only a local minimum of the free energy. The differences between this phase diagram and the equivalent one for the random bond ATM in the Bethe lattice are essentially the same discussed by Moreira and Christiano (1991).

Case D. $J_3^0 = J = 0$ (figure 1(*d*))

In this case, the phase diagram is quite similar to those obtained in the Sherrington–Kirpatrick version and in the Bethe lattice. It differs only by the presence of a F–II phase in which the individual spins order ferromagnetically and the pair σS 'orders' in a mixed phase. This is a quite unexpected situation because the ferromagnetic ordering of the individual spins should determine at least in principle the ferromagnetic ordering of the pair σS. The appearance of this F–II phase seems to be a consequence of the great independence between the order parameters, as discussed in case A. In both regions D and E the phases F–II and F–F coexist but while in D the phase F–II represents the global minimum of the free energy, in E this minimum corresponds to the phase F–F.

References

Amit D J, Gutfreund H and Sompolinsky H 1985 *Phys. Rev.* A **32** 1007
Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178
Christiano P L and Goulart Rosa S 1985 *Phys. Lett.* **110A** 44
—— 1986 *Phys. Rev.* A **34** 730
Fan C 1972 *Phys. Lett.* **39A** 136
Hopfield J J 1982 *Proc. Natl Acad. Sci. USA* **79** 2554
Kirpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384
Moreira J V and Christiano P L 1991 *Phys. Lett.* A in press
Provost J P and Vallee G 1983 *Phys. Rev. Lett.* **50** 598
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
van Hemmen J L 1982 *Phys. Rev. Lett.* **49** 409
van Hemmen J L, van Enter A C D and Canisius J 1983 *Z. Phys.* B **50** 311